A Non-asymptotic Framework for the Approximate Message Passing Algorithm

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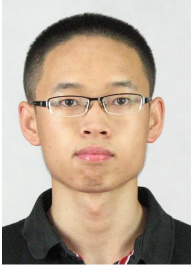
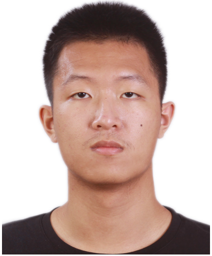


Gen Li, UPenn

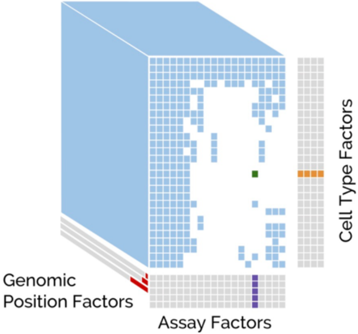


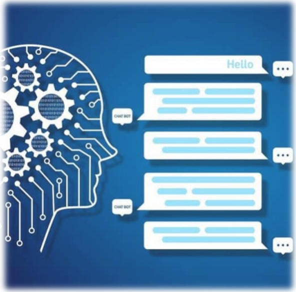
Wei Fan, UPenn

"*A non-asymptotic framework for approximate message passing in spiked models,*"
Gen Li, Yuting Wei, *arxiv.2208.03313*

"*Approximate message passing from random initialization with applications to* $\mathbb{Z}_2$ *synchronization,*" Gen Li, Wei Fan, Yuting Wei, *arxiv.2302.03682*

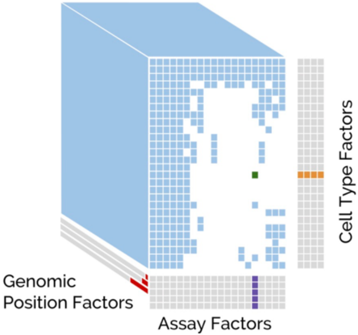
High-dimensional statistical tasks

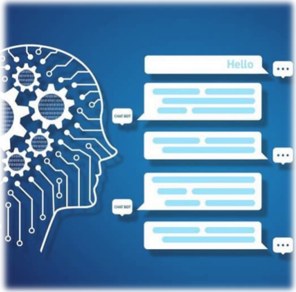




Statistical tasks: linear regression, generalized linear models, low-rank matrix estimation, phase retrieval, tensor decomposition...

High-dimensional statistical tasks



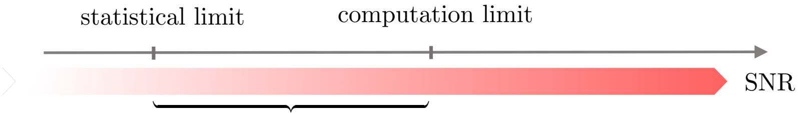


Statistical tasks: linear regression, generalized linear models, low-rank matrix estimation, phase retrieval, tensor decomposition...

When problem sizes are large, computation complexity is an issue!

Statistical accuracy vs. computation complexity

statistical-to-computational gap in problems with combinatorial nature *(e.g. community detection, planted cliques, sparse principal component analysis, structured matrix models, sparse tensor models...)*



"I can't find an efficient algorithm, but neither can all these people."

— *see survey Bandeira, Perry, Wein'18*

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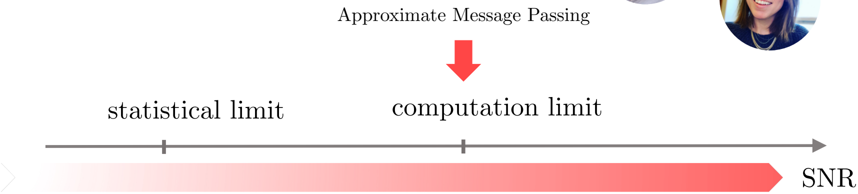
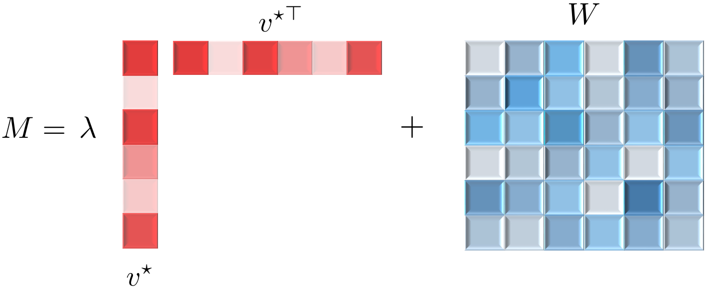


— *see tutorial Feng, Venkataramanan, Rush, Samworth' 22*

A simple model: spiked Wigner model



$$M = \lambda v^{\star} v^{\star\top} + W$$

Johnstone (2001),

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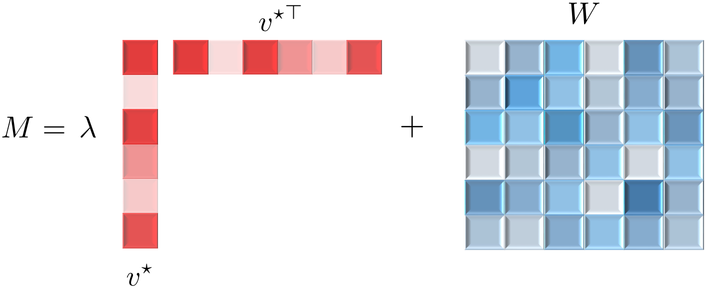

- $W_{ij} = W_{ji} \sim \mathcal{N}(0, \frac{1}{n})$ and $W_{ii} \sim \mathcal{N}(0, \frac{2}{n})$

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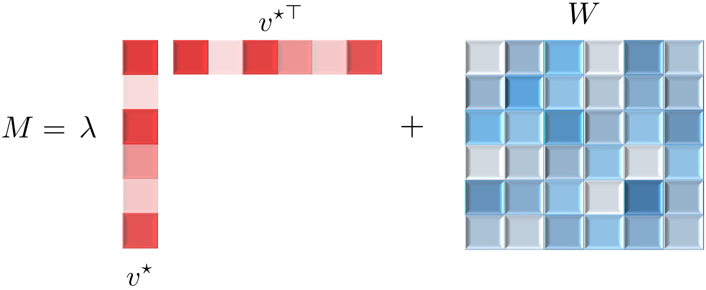

- $W_{ij} = W_{ji} \sim \mathcal{N}(0, \frac{1}{n})$ and $W_{ii} \sim \mathcal{N}(0, \frac{2}{n})$
- $\lambda =$ SNR (signal-to-noise ratio) with $\|v^{\star}\|_2 = 1$
- Goal: estimate $v^{\star}$ from M

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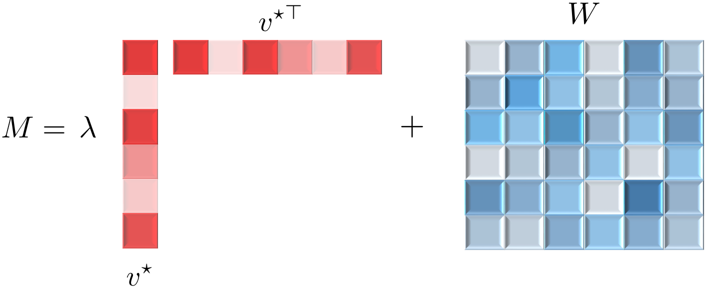

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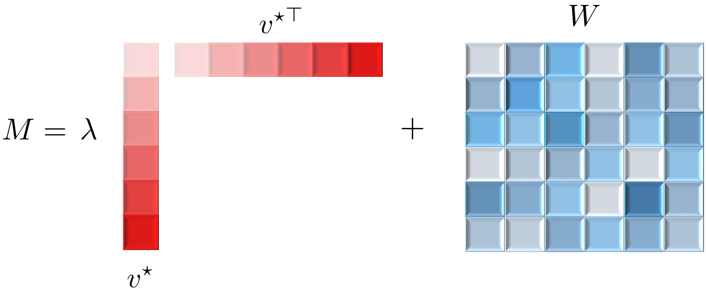
- $\lambda =$ SNR (signal-to-noise ratio) with $\|v^\star\|_2 = 1$
- Goal: estimate $v^\star$ from M
- Phase transition at $\lambda > 1$: the top eigenvalue separates from bulk, eigenvector correlates non-trivially with $v^\star$

Johnstone (2001), Johnstone & Lu (2004), Péché (2006), Baik & Silverstein (2006), Capitaine, Donati-Martin & Féral (2009), Féral & Péché (2007)...

Spiked Wigner model with structures



Applications: spin-glass problems, community detection, image alignment, angular synchronization

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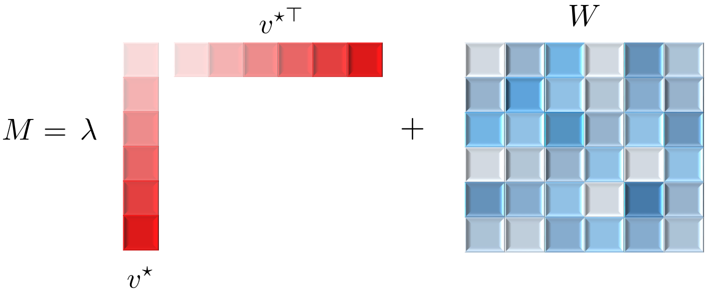

Applications: spin-glass problems, community detection, image alignment, angular synchronization

- $\mathbb{Z}_2$ synchronization: $\sqrt{n}v_i^{\star} \overset{\text{i.i.d.}}{\sim} \text{Unif}\{+1, -1\}$

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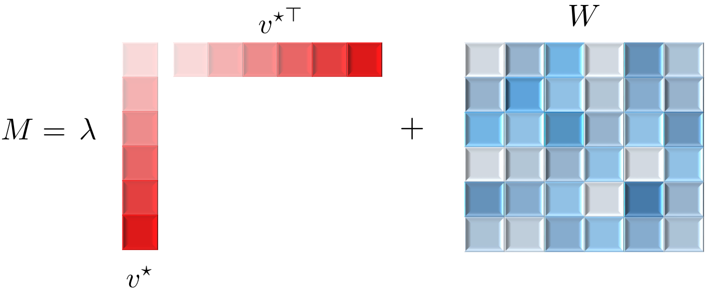

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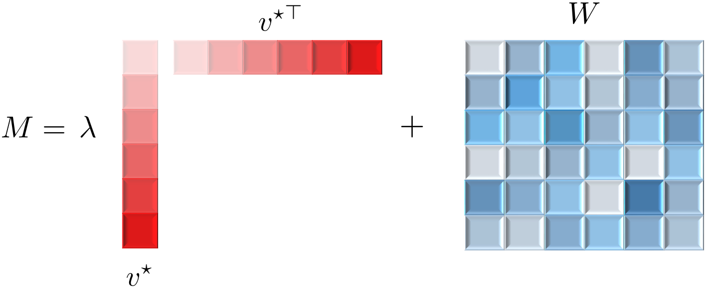
- sparse Wigner model: $\|v^\star\|_0 = k$

Singer (2011), Panchenko (2013), Deshpande, Abbe & Montanari (2016), Perry, Wein, Bandeira, Moitra (2018), Javanmard, Montanari & Ricci-Tersenghi (2016)...

Spiked Wigner model with structures



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- non-negative Wigner model: $v_i^\star \geq 0$

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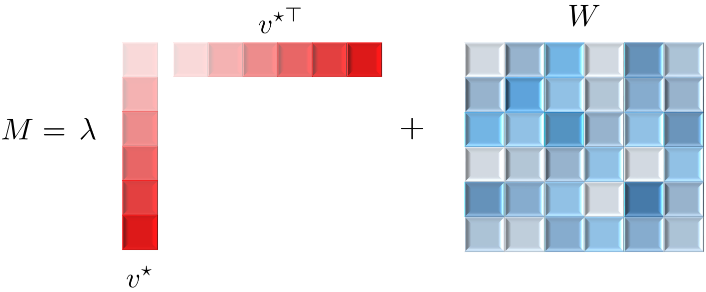

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- sparse Wigner model: $\|v^\star\|_0 = k$
- non-negative Wigner model: $v_i^\star \geq 0$
- cone-constrained spiked models: $v^\star \in \mathcal{K}$ (e.g. monotone, convex)

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An incomplete list of prior art

$\mathbb{Z}_2$ synchronization:

- Baik, Arous, Péché'05
- Panchenko'13
- Javanmard et al.'16
- Montanari & Sen'16
- Lelarge & Miolane'19
- Deshpande, Abbe, Montanari'17
- Celentano, Fan, Mei'21

general convex cones:

- Deshpande, Montanari, Richard'14
- Lesieur, Krzakala, Zdeborová'17
- Bandeira, Kunisky, Wein'19

sparse PCA (Wigner / Wishart)

- Johnstone & Lu'09
- d'Aspremont et al.'04
- Amini & Wainwright'08
- Vu & Lei'12
- Berthet & Rigollet'13
- Ma'13
- Lesieur, Krzakala, Zdeborová'15
- Deshpande & Montanari'14
- Wang, Berthet, Samworth'16
- Ding, Kunisky, Wein, Bandeira'19

positive Wigner models

- Montanari & Richard'16

Idealistic estimators

$$\text{Maximum likelihood estimator} \coloneqq \arg \min_{\substack{v \in \mathcal{S}^{n-1} \\ v \text{ with structures}}} \|M - \lambda v v^\top\|_F^2$$

$$\text{Bayes optimal estimator} \coloneqq \mathbb{E}[v v^\top \mid M]$$

AMP for spiked models

Maximum likelihood estimator $:= \arg \min_{\substack{v \in \mathcal{S}^{n-1} \\ v \text{ with structures}}} \|M - \lambda v v^\top\|_F^2$

Bayes optimal estimator $:= \mathbb{E}[v v^\top \mid M]$

— in general, computationally infeasible...

AMP for spiked models

Approximate message passing (AMP) for spiked models:

$$x_{t+1} = M\eta_t(x_t) - \langle \eta_t'(x_t) \rangle \cdot \eta_{t-1}(x_{t-1}), \text{ for } t \geq 1$$

where $\langle x \rangle := \frac{1}{n}\sum_{i=1}^{n} x_i$.

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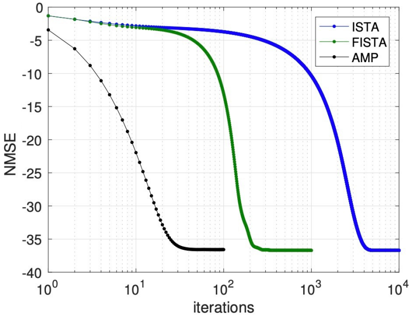
- Onsager correction term $\langle \eta_t'(x_t) \rangle \cdot \eta_{t-1}(x_{t-1})$
- η_t: denoising function selected *a priori* (tailored to structure of $v^\star$)
  - $\mathbb{Z}_2$ **synchronization:** $\eta_t(x) = \rho_t \tanh(x)$
  - **sparse estimation:** $\eta_t(x) = \rho_t \cdot \mathrm{sign}(x)(|x| - \tau_t)_+$
  - **general cone:** $\eta_t(x) = \rho_t \cdot \mathrm{Proj}_{\mathcal{K}}(x)$

Some background of AMP

- AMP is a low-complexity, iterative algorithm
  [Donoho, Maleki, Montanari (2009, 2010a, 2011b), Bayati & Montanari (2011)...]



AMP in computing LASSO

Some background of AMP

- AMP is a low-complexity, iterative algorithm
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- Theoretically optimal vs. computationally feasible estimators
  [Reeves, Pfister (2019), Barbier et al. (2017), Lelarge & Miolane (2019), Montanari & Ramji (2019), Celentano & Montanari (2019)...]

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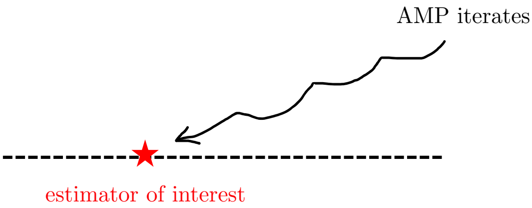
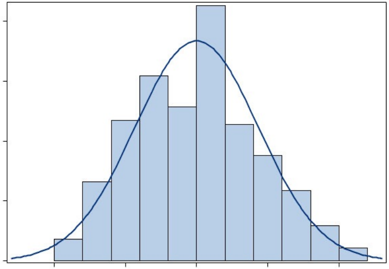
- A useful tool to analyze other statistical procedures [Donoho, Maleki, Montanari (2009), Donoho & Montanari (2016), Sur, Chen, Candès. (2017), Bu et al. (2020), Fan & Wu (2021), Li & Wei (2021)...]



Prior theory of AMP

Exact asymptotics: for constant # iterations t (e.g. $t = 20$), empirical distribution of the coordinates of AMP iterate $x_t \in \mathbb{R}^n$ is approximately Gaussian ($n \to \infty$)



histogram of coordinates of x_t

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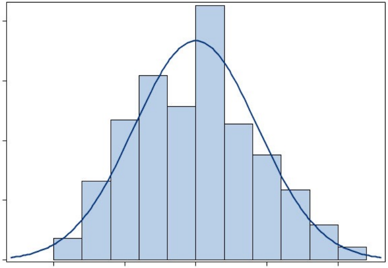

histogram of coordinates of x_t

Its variance is given by low-dimensional recursion:

state evolution: $\tau_{t+1} = F(\tau_t)$

τ_t captures the variance at iteration t

[Bayati & Montanari (2011), Javanmard & Montanari (2013), Schniter & Rangan (2014)]

Prior results: exact asymptotics

Theorem (Montanari & Venkataramanan'19)

Suppose the empirical distribution $\{v_i^\star\}_{i=1}^n \to \mu_V$ *on* $\mathbb{R}$, *with* $\mathbb{E}[V^2] = 1$. *For constant # iterations* t *(independent of* n*), it satisfies,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (x_{t,i} - v_i^\star)^2 = \mathbb{E}\Big[\big(\alpha_t V + \beta_t G - V\big)^2\Big], \qquad \text{a.s.}$$

where $V \sim \mu_V$ *and* $G \sim \mathcal{N}(0,1)$ *are independent.*

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where $V \sim \mu_V$ *and* $G \sim \mathcal{N}(0,1)$ *are independent.*

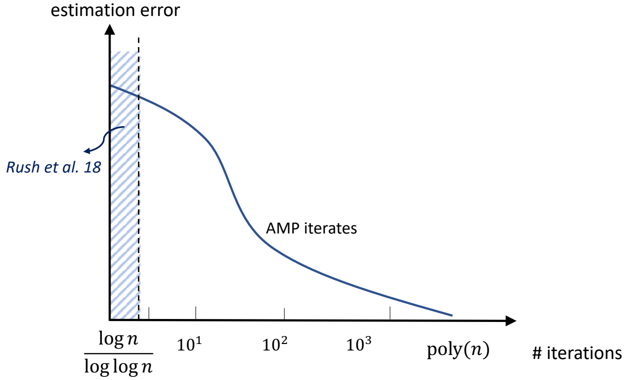
- State evolution (SE) via the recursion

$$(\alpha_{t+1}, \beta_{t+1}) = F(\alpha_t, \beta_t) = \begin{cases} \alpha_{t+1} = \lambda \mathbb{E}\big[V \cdot \eta_t(\alpha_t V + \beta_t G)\big] \\ \beta_{t+1}^2 = \mathbb{E}\big[\eta_t^2(\alpha_t V + \beta_t G)\big] \end{cases}$$

Non-asymptotic analyses are quite limited so far...

- compared to other optimization methods
- compared to other analysis techniques

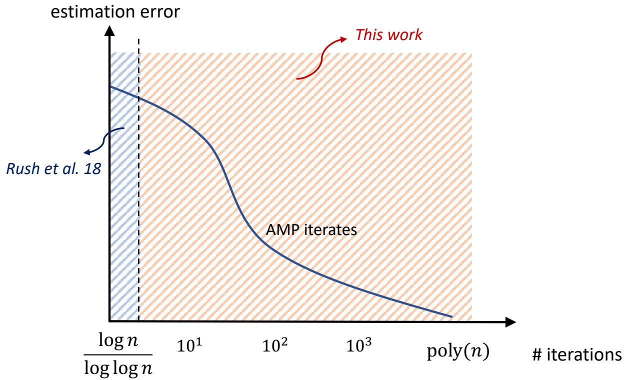
Non-asymptotic analysis?



Non-asymptotic result: Rush & Venkataramanan (2018)

$\#\text{iterations} = o(\log n / \log\log n)$ (*based on state-evolution analysis*)

Non-asymptotic analysis?



Question: Is it possible to develop non-asymptotic analysis of AMP beyond $o(\log n/ \log\log n)$ iterations?

Our solution: *a new decomposition for AMP iterates*

This work: a new decomposition of AMP

Theorem (Li & Wei'22)

Initialize AMP with x_1 *independent of* W*. For every* $1 \le t \le n$*, AMP yields the decomposition*

$$x_{t+1} = \alpha_{t+1} v^\star + \sum_{k=1}^{t} \beta_t^k \phi_k + \xi_t, \qquad (\star)$$

for $\phi_k \overset{i.i.d.}{\sim} \mathcal{N}(0, \frac{1}{n} I_n)$.

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here $(\alpha_{t+1}, \beta_t, \xi_t)$ obeys

$$\alpha_{t+1} = \lambda {v^\star}^\top \eta_t(x_t),$$

$$\beta_t^k = \langle \eta_t(x_t), z_k \rangle \qquad \text{for an explicit-defined basis } \{z_k\}$$

$$\|\xi_t\|_2 = \Big\langle \sum_{k=1}^{t-1} \mu^k \phi_k, \delta_t \Big\rangle - \langle \delta_t' \rangle \sum_{k=1}^{t-1} \mu^k \beta_{t-1}^k + \Delta_t + O\Big(\sqrt{\frac{t \log n}{n}} \|\beta_t\|_2\Big) \quad \text{w.h.p.}$$

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- x_t behaves like $\alpha_t v^\star + \sum_{k=1}^{t-1} \beta_{t-1}^k \phi_k$ if $\|\xi_{t-1}\|_2$ is small

$$\text{Wasserstein}_1 \left(\mu\Big(\frac{1}{\|\beta_{t-1}\|_2} \sum_{k=1}^{t-1} \beta_{t-1}^k \phi_k \Big),\ \mathcal{N}\big(0, \frac{1}{n} I_n\big) \right) \le \sqrt{\frac{t \log n}{n}}.$$

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- if $\{\eta_t\}$ are nice (*smooth & with finite jumps*), we can track how $\|\xi_t\|_2$ depends on λ, t, n

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- if $\{\eta_t\}$ are nice (*smooth & with finite jumps*), we can track how $\|\xi_t\|_2$ depends on λ, t, n
- decomposition $(\star)$ can be extended for spectral initialization

Finite-sample error control

Theorem (Li & Wei'22 (informal))

AMP iterates satisfy $x_{t+1} = \alpha_{t+1} v^\star + \sum_{k=1}^{t} \beta_t^k \phi_k + \xi_t$ *w.h.p. with*

$$\alpha_{t+1} = \lambda v^{\star\top} \int \eta_t \left(\alpha_t v^\star + \frac{1}{\sqrt{n}} x \right) \varphi_n(\mathrm{d}x) + \lambda \Delta_{\alpha,t}, \quad \|\beta_t\|_2 = 1,$$

where the residual terms obey

$$|\Delta_{\alpha,t}| \lesssim B_t + \rho \|\xi_{t-1}\|_2,$$

$$\|\xi_t\|_2 \le \kappa_t \|\xi_{t-1}\|_2 + O\left(A_t + \rho \sqrt{\frac{t \log n}{n}} \|\xi_{t-1}\|_2 \right).$$

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It suffices to control

- $\kappa_t < 1 - c$
- A_t corresponds to an upper bound for quantity

$$\Big| \sum_{k=1}^{t-1} \mu^k \Big[\underbrace{\langle \phi_k, \eta_t(v_t) \rangle - \langle \eta_t'(v_t) \rangle \beta_{t-1}^k}_{Y_k} \Big] \Big|, \qquad \text{with } v_t := \alpha_t v^\star + \sum_{k=1}^{t-1} \beta_{t-1}^k \phi_k$$

Application in a concrete example: $\mathbb{Z}_2$ synchronization

Prior art: A hybrid procedure

- Setting: $M = \lambda v^{\star} v^{\star\top} + W$ where $\sqrt{n} v_i^{\star} \sim \mathsf{Unif}(\{\pm 1\})$
- Goal: recover $v^{\star}$ given M

— *AMP is approximately Gaussian in a fixed t, large n limit*

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A hybrid procedure proposed in Celentano, Fan, Mei'21

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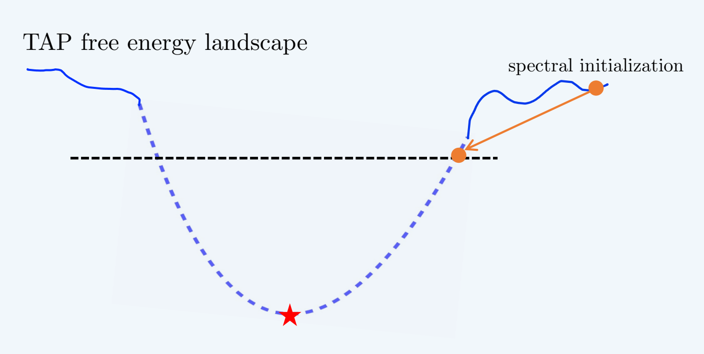

Prior art: A hybrid procedure

- Setting: $M = \lambda v^\star v^{\star\top} + W$ where $\sqrt{n} v_i^\star \sim \text{Unif}(\{\pm 1\})$
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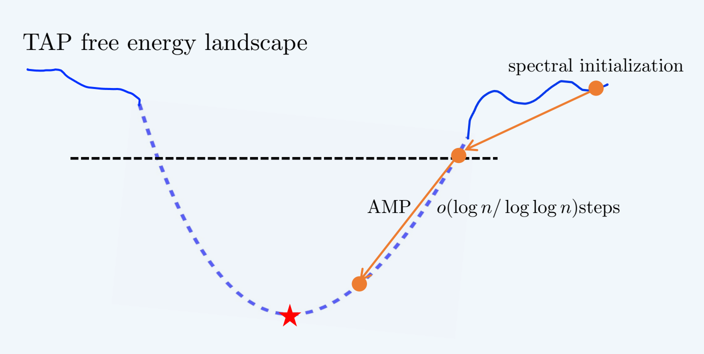

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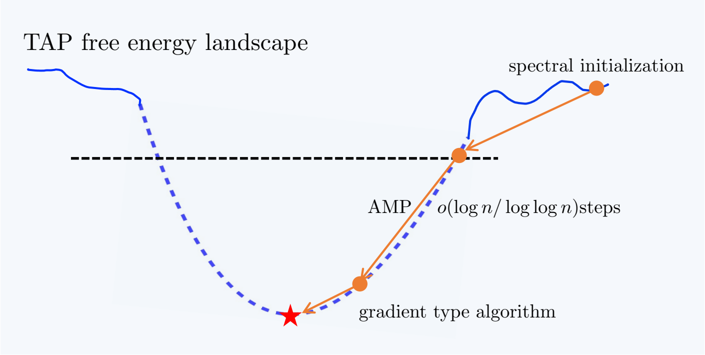

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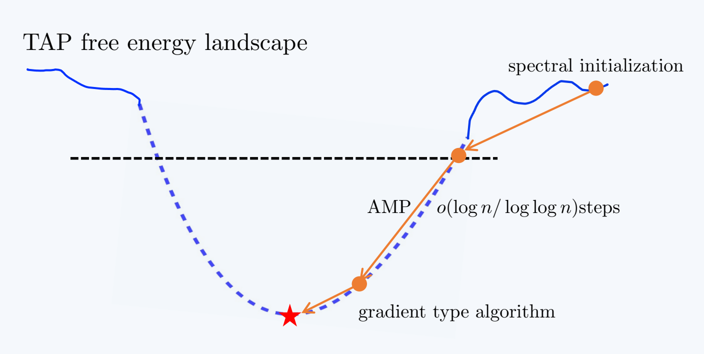


Open question: spectrally-initialized AMP is sufficient for $\lambda > 1$?

$\mathbb{Z}_2$ Synchronization: our results

Theorem (Li & Wei'22)

Spectrally-initialized AMP satisfies

$$x_{t+1} = \alpha_{t+1} v^\star + \sum_{k=1}^{t} \beta_t^k \phi_k + \xi_t,$$

with

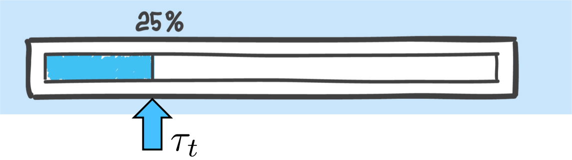
$$\alpha_{t+1} = \mathbb{E}\left[\lambda v^{\star\top} \eta_t\Big(\alpha_t v^\star + \frac{1}{\sqrt{n}} G\Big)\right] + O\left(\sqrt{\frac{t \log n}{(\lambda-1)^3 n}}\right),$$

$$\|\beta_t\|_2 = 1, \qquad \|\xi_t\|_2 \lesssim O\left(\sqrt{\frac{t \log n}{(\lambda-1)^3 n}} + \sqrt{\frac{\log^7 n}{(\lambda-1)^9 n}}\right)$$

w.h.p. provided that $t \lesssim \frac{(\lambda-1)^{10}}{\log^7 n} n.$

- spectral initialization provides a warm-start with $\alpha_1 \asymp \sqrt{\lambda^2 - 1}$

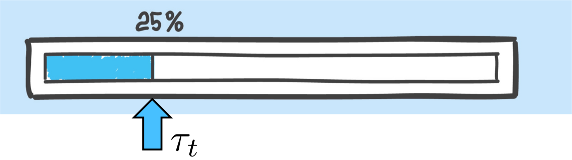
Connection to state evolution



(asymptotic) state evolution Deshpande, Abbe, Montanari (2016):

$$\tau_{t+1} := \lambda^2 \int \tanh(\tau_t + \sqrt{\tau_t} x) \varphi(\mathrm{d}x)$$

Connection to state evolution



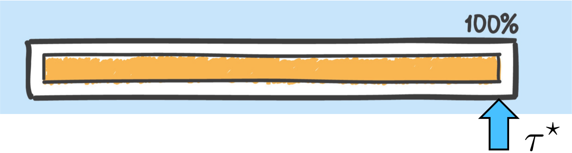
(asymptotic) state evolution Deshpande, Abbe, Montanari (2016):

$$\tau_{t+1} := \lambda^2 \int \tanh(\tau_t + \sqrt{\tau_t} x) \varphi(\mathrm{d}x)$$

here

$$\alpha_t^2 - \tau_t = O\left(\sqrt{\frac{t \log n}{(\lambda - 1)^8 n}} + \sqrt{\frac{\log^7 n}{(\lambda - 1)^{14} n}} \right)$$

Connection to state evolution

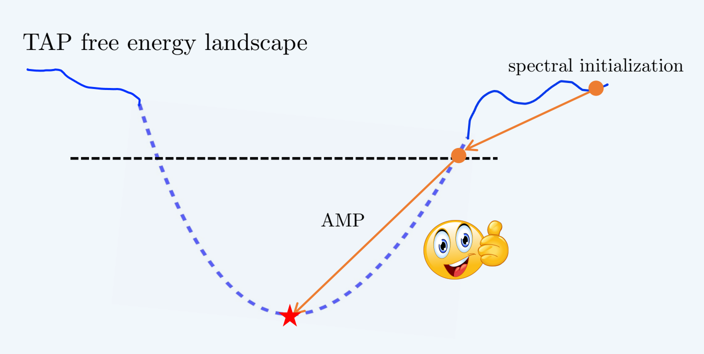


(asymptotic) state evolution Deshpande, Abbe, Montanari (2016):

$$\tau_{t+1} \coloneqq \lambda^2 \int \tanh(\tau_t + \sqrt{\tau_t} x)\varphi(\mathrm{d}x)$$

$$\alpha_t^2 - \tau^\star = c\big(1 - (\lambda - 1)\big)^t + O\left(\sqrt{\frac{t \log n}{(\lambda-1)^8 n}} + \sqrt{\frac{\log^7 n}{(\lambda-1)^{14} n}}\right)$$

Take-home message #1



- Answer the open question (Celentano, Fan & Mei (2021)) positively: spectrally-initialized AMP is enough!

$\mathbb{Z}_2$ Synchronization: simulations

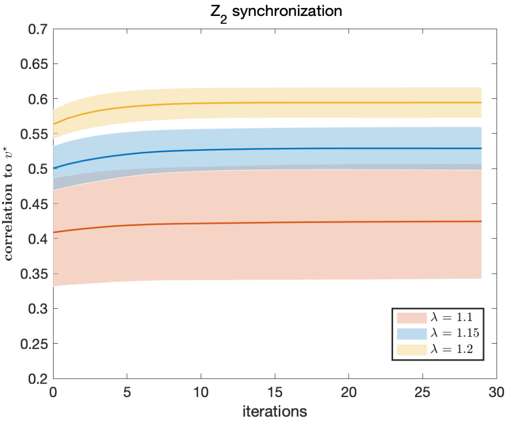


Figure: Convergence of spectrally-initialized AMP for different signal strengths with $n = 10000$. Repeat 40 times.

Question: *Is spectral initialization really necessary for AMP?*

Similation: AMP with random initialization

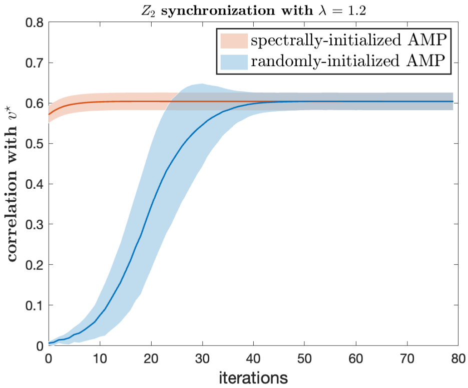


Figure: The correlation of $\eta_t(x_t)$ and $v^\star$ vs. iteration count t for AMP with both random and spectral initialization. Here $n = 10000$. Repeat 20 times.

AMP with random initialization

Theorem (Li, Fan, Wei'23)

For $t \le \frac{cn(\lambda-1)^5}{\log^2 n}$, randomly-initialized AMP satisfies, w.h.p,

- **(Decomposition)** $x_{t+1} = \alpha_{t+1} v^\star + \sum_{k=1}^{t} \beta_t^k \phi_k + \xi_t$, *with*

$$\alpha_{t+1} \coloneqq \lambda v^{\star\top} \eta_t(x_t),$$

$$\|\beta_t\|_2 = 1, \quad \|\xi_t\|_2 \lesssim \sqrt{\frac{t \log n}{n(\lambda-1)^2}} + \sqrt{\frac{\log^4 n}{n(\lambda-1)^3}};$$

— $\tau_{t+1} \coloneqq \lambda^2 \int \tanh(\tau_t + \sqrt{\tau_t} x) \varphi(\mathrm{d}x)$

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- **(Crossing time)**

$$\varsigma := \min\{t : |\alpha_t| \geq \frac{1}{2}\sqrt{\lambda^2 - 1}\} = O\left(\frac{\log n}{\lambda - 1}\right);$$

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AMP with random initialization

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$$\alpha_{t+1} := \lambda {v^\star}^\top \eta_t(x_t),$$

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- **(Crossing time)**

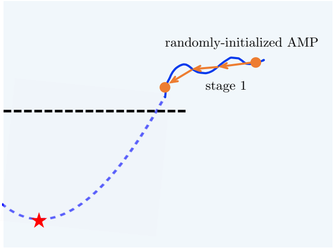
$$\varsigma := \min\{t : |\alpha_t| \geq \frac{1}{2}\sqrt{\lambda^2 - 1}\} = O\left(\frac{\log n}{\lambda - 1}\right);$$

- **(Non-asymptotic SE)** *for any* $t \geq \varsigma$,

$$\alpha_t^2 = \left(1 + O\left(\sqrt{\frac{\left(t + \frac{\log^3 n}{\lambda-1}\right)\log n}{n(\lambda-1)^5}}\right)\right)\tau_{t+1}.$$

— $\tau_{t+1} := \lambda^2 \int \tanh(\tau_t + \sqrt{\tau_t} x)\varphi(\mathrm{d}x)$

Dynamics after random initialization



randomly-initialized AMP

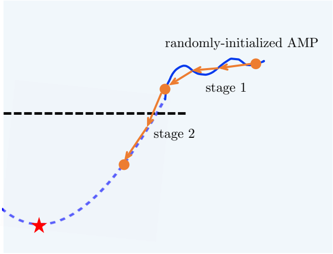
- escape from random initialization

$$\alpha_{t+1} \approx \lambda \alpha_t + \lambda g_{t-1}$$

$\alpha_t \approx n^{-1/4}$

$O\left(\frac{\log n}{\lambda - 1}\right)$ #steps

Dynamics after random initialization



randomly-initialized AMP

- escape from random initialization

$$\alpha_{t+1} \approx \lambda \alpha_t + \lambda g_{t-1}$$

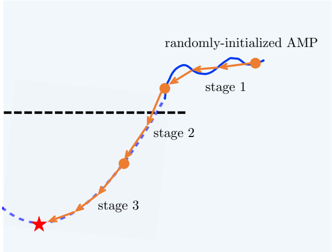
- exponential growth

$$\alpha_{t+1} \geq \big(1 + c(\lambda - 1)\big)^{1/2} \, \alpha_t$$

$\alpha_t \approx n^{-1/4}$ $\qquad$ $\alpha_t \approx \sqrt{\lambda^2 - 1}$

$O\Big(\frac{\log n}{\lambda - 1}\Big)$ #steps $\qquad$ $O\Big(\frac{\log n}{\lambda - 1}\Big)$ #steps

Dynamics after random initialization



randomly-initialized AMP

- escape from random initialization

$\alpha_{t+1} \approx \lambda \alpha_t + \lambda g_{t-1}$

- exponential growth

$\alpha_{t+1} \geq \big(1 + c(\lambda - 1)\big)^{1/2} \alpha_t$

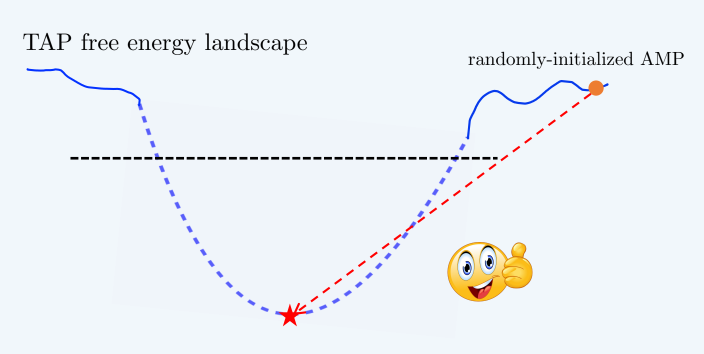
- local refinement

$|\alpha_t^2 - \tau^\star| \lesssim \big(1 - (\lambda - 1)\big)^{t-\varsigma} + \sqrt{\frac{t/n}{(\lambda - 1)^6}}$

$\alpha_t \approx n^{-1/4}$ $\qquad$ $\alpha_t \approx \sqrt{\lambda^2 - 1}$

$O\Big(\frac{\log n}{\lambda - 1}\Big)$ #steps $\qquad$ $O\Big(\frac{\log n}{\lambda - 1}\Big)$ #steps $\qquad$ $t \leq \frac{n(\lambda - 1)^5}{\log^2 n}$

Take-home message #2



- It takes *randomly-initialized* AMP at most $O\big(\frac{\log n}{\lambda-1}\big)$ iterations to get $\widetilde{O}\big(\sqrt{\frac{1}{n(\lambda-1)^6}}\big)$ close to the Bayes-optimal risk.

A glimpse of our main proof idea...

— *decomposition:* $x_{t+1} = \alpha_{t+1} v^\star + \sum_{k=1}^{t} \beta_t^k \phi_k + \xi_t$

Prior non-asymptotic guarantees

AMP for spiked models:

$$x_{t+1} = M\eta_t(x_t) - \langle \eta_t'(x_t) \rangle \cdot \eta_{t-1}(x_{t-1}), \text{ for } t \geq 1$$

- **Challenges:** deal with statistical dependence between iterations

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- **Challenges:** deal with statistical dependence between iterations
- Rush & Venkataramanan'16 $\#$iterations $= o(\log n / \log\log n)$
  — *based on state-evolution analysis in Bayati & Montanari'11*

statistical dependence   induction step

$$\mathbb{P}(\text{residual at time } t \geq \epsilon)$$

$$= \mathbb{P}\left(\sum_{i=0}^{t-1} r_i^t \geq \epsilon\right) \leq \sum_{i=0}^{t-1} \mathbb{P}\left(r_i^t \leq \frac{\epsilon}{t}\right) \leq tC_{t-1} \exp\left(-\frac{c_{t-1}}{t^2} n\epsilon^2\right)$$

requires $\frac{n}{(t!)^2} \to \infty \to t = o(\log n / \log\log n)$

Main proof idea: a new decomposition

- $\{x_t\}$ is the sequence generated by AMP

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- $\{x_t\}$ is the sequence generated by AMP
- define an orthonormal basis $\{z_t\}$ where

$$z_1 := \frac{\eta_1(x_1)}{\|\eta_1(x_1)\|_2} \in \mathbb{R}^n, \qquad \text{and} \qquad W_1 := W$$

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- write $U_{t-1} \coloneqq [z_k]_{1\le k\le t-1} \in \mathbb{R}^{n\times(t-1)}$ and denote

$$z_t \coloneqq \frac{\left(I - U_{t-1}U_{t-1}^\top\right)\eta_t(x_t)}{\left\|\left(I - U_{t-1}U_{t-1}^\top\right)\eta_t(x_t)\right\|_2} \qquad \textit{Gram–Schmidt orthogonalization},$$

$$W_t \coloneqq \left(I - z_{t-1}z_{t-1}^\top\right)W_{t-1}\left(I - z_{t-1}z_{t-1}^\top\right)$$

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$$W_t := \left(I - z_{t-1}z_{t-1}^\top\right) W_{t-1} \left(I - z_{t-1}z_{t-1}^\top\right)$$

- write $\eta_t(x_t) = \sum_{k=1}^{t} \beta_t^k z_k$, for $\beta_t^k := \langle \eta_t(x_t), z_k \rangle$

Main proof idea: a new decomposition

- AMP updates:

$$x_{t+1} = M\eta_t(x_t) - \langle \eta_t'(x_t) \rangle \cdot \eta_{t-1}(x_{t-1}), \text{ where } M = \lambda v^\star v^{\star\top} + W$$

- Goal: $x_{t+1} = \alpha_{t+1} v^\star + \sum_{k=1}^{t} \beta_t^k \phi_k + \xi_t$

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$M\eta_t(x_t)$

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$M\eta_t(x_t)$

$$= v^\star \underbrace{\lambda v^{\star\top}\eta_t(x_t)}_{\alpha_{t+1}} + \Big\{ W_t + \sum_{k=1}^{t-1} \underbrace{\Big[W_k z_k z_k^\top + z_k z_k^\top W_k - z_k z_k^\top W_k z_k z_k^\top \Big]}_{W_k - W_{k+1}} \Big\} \cdot \underbrace{\sum_{k=1}^{t} \beta_t^k z_k}_{\eta_t(x_t)}$$

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$$= \alpha_{t+1} v^\star + \sum_{k=1}^{t} \beta_t^k W_k z_k + \dots$$

$$= \alpha_{t+1} v^\star + \sum_{k=1}^{t} \beta_t^k \underbrace{(W_k z_k + \zeta_k)}_{\phi_k \sim \mathcal{N}(0, \frac{1}{n} \mathbf{I}_n)} + \dots$$

Main proof idea: a new decomposition

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- Goal: $x_{t+1} = \alpha_{t+1} v^\star + \sum_{k=1}^{t} \beta_t^k \phi_k + \xi_t$

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$$= v^\star \underbrace{\lambda v^{\star\top} \eta_t(x_t)}_{\alpha_{t+1}} + \Big\{ W_t + \sum_{k=1}^{t-1} \underbrace{\Big[W_k z_k z_k^\top + z_k z_k^\top W_k - z_k z_k^\top W_k z_k z_k^\top \Big]}_{W_k - W_{k+1}} \Big\} \cdot \underbrace{\sum_{k=1}^{t} \beta_t^k z_k}_{\eta_t(x_t)}$$

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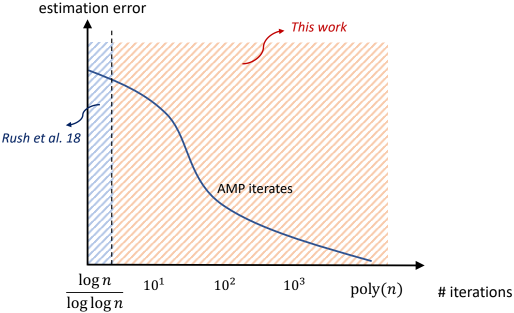
$$= \alpha_{t+1} v^\star + \sum_{k=1}^{t} \beta_t^k \underbrace{(W_k z_k + \zeta_k)}_{\phi_k \sim \mathcal{N}(0, \frac{1}{n} \mathbf{I}_n)} + \dots$$

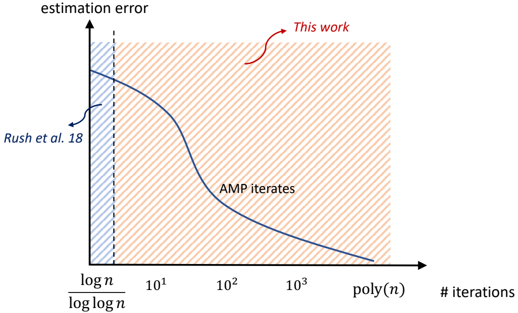
$$\xi_t = \sum_{k=1}^{t-1} z_k \Big[\langle W_k z_k, \eta_t(x_t) \rangle - \langle \eta_t'(x_t) \rangle \beta_{t-1}^k - \beta_t^k z_k^\top W_k z_k \Big] - \sum_{k=1}^{t} \beta_t^k \zeta_k$$

Concluding remarks



- a new non-asymptotic framework of AMP that allows for # iterations $O(\frac{n}{\text{poly}(\log n)})$ given informative/spectral initialization

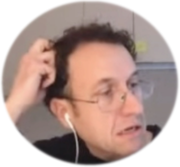
Concluding remarks



- a new non-asymptotic framework of AMP that allows for # iterations $O(\frac{n}{\mathrm{poly}(\log n)})$ given informative/spectral initialization
- analyze performance of randomly-initialized AMP for $\mathbb{Z}_2$ synchronization

Concluding remarks: future extensions

- other statistical settings
- AMP for non-separable denoising functions
- connections to other polynomial-time algorithms
- universality results
- infinite number of iterations
- etc...



This is probably all trivial ...

Thanks for your attention! Questions?

Paper:

"A non-asymptotic framework for approximate message passing in spiked models," G. Li, Y. Wei, *arxiv.2208.03313*

"Approximate message passing from random initialization with applications to $\mathbb{Z}_2$ synchronization," G. Li, W. Fan, Y. Wei, *arxiv.2302.03682*

"Non-asymptotic analyses for approximate message passing with applications to sparse and robust regression," G. Li, Y. Wei, *upcoming*

sparse PCA in spiked models

- Setting: $M = \lambda v^{\star} v^{\star\top} + W$ where $\|v^{\star}\|_0 = k$
- Goal: recover $v^{\star}$ given M

$\lambda \approx \sqrt{\frac{k \log n}{n}}$

statistical limit

$\lambda \approx \sqrt{\frac{k^2}{n}}$

reduction to planted cliques:
Berthet & Rigollet (2013)

computation limit

SNR

"I can't find an efficient algorithm, but neither can all these people."

Zou et al. (2006)
Amini and Wainwright (2008)
Ma (2013)
Deshpande and Montanari (2014b)
Hopkins et al. (2017)

Sparse PCA: our results

Theorem (Li & Wei'22)

Suppose $0 < \lambda \lesssim 1$. *Given an informative initialization (with non-vanishing correlation with* $v^\star$*), AMP satisfies*

$$x_{t+1} = \alpha_{t+1} v^\star + \sum_{k=1}^{t} \beta_t^k \phi_k + \xi_t,$$

with

$$\alpha_{t+1} = \mathbb{E}\left[\lambda v^{\star\top} \eta_t\Big(\alpha_t v^\star + \frac{1}{\sqrt{n}} G\Big)\right] + \sqrt{\frac{k + t\log^3 n}{n}},$$

$$\|\beta_t\|_2 = 1, \qquad \|\xi_t\|_2 \lesssim \sqrt{\frac{k + t\log^3 n}{n}} \qquad \textit{w.h.p.}$$

provided that $\frac{t\log^3 n}{n\lambda^2} \lesssim 1$ *and* $\frac{k\log n}{n\lambda^2} \lesssim 1$.

Sparse PCA: our results

Theorem (Li & Wei'22)

Suppose $0 < \lambda \lesssim 1$. *Given an informative initialization (with non-vanishing correlation with* $v^\star$*), AMP satisfies*

$$x_{t+1} = \alpha_{t+1} v^\star + \sum_{k=1}^{t} \beta_t^k \phi_k + \xi_t,$$

with

$$\alpha_{t+1} = \mathbb{E}\left[\lambda v^{\star\top} \eta_t\Big(\alpha_t v^\star + \frac{1}{\sqrt{n}} G\Big)\right] + \sqrt{\frac{k + t\log^3 n}{n}},$$

$$\|\beta_t\|_2 = 1, \qquad \|\xi_t\|_2 \lesssim \sqrt{\frac{k + t\log^3 n}{n}} \qquad \textit{w.h.p.}$$

provided that $\frac{t\log^3 n}{n\lambda^2} \lesssim 1$ *and* $\frac{k\log n}{n\lambda^2} \lesssim 1$.

denoising functions:

$$\eta_t(x) = \gamma_t \mathrm{sign}(x)(|x| - \tau_t)_+ \quad \text{where } \gamma_t^{-1} := \|(|x_t| - \tau_t)_+\|_2, \tau_t \asymp \sqrt{\frac{\log n}{n}}$$

Several remarks

- recall the (asymptotic) state evolution:

$$\alpha_{t+1}^{\star} := \frac{\lambda v^{\star\top} \int \mathsf{ST}_{\tau_t}\left(\alpha_t^{\star} v^{\star} + \frac{x}{\sqrt{n}}\right) \varphi_n(\mathrm{d}x)}{\sqrt{\int \left\|\mathsf{ST}_{\tau_t}\left(\alpha_t^{\star} v^{\star} + \frac{x}{\sqrt{n}}\right)\right\|_2^2 \varphi_n(\mathrm{d}x)}}$$

  then

$$\left|\alpha_{t+1} - \alpha_{t+1}^{\star}\right| \lesssim \sqrt{\frac{k \log n + t \log^3 n}{n}}$$

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- two sufficient initialization schemes:
  - AMP with diagonal maximization: $\lambda \|v^{\star}\|_{\infty} \gtrsim \sqrt{\frac{k \log n}{n}}$
  - AMP with sample-split initialization: $\lambda \gtrsim \sqrt{\frac{k^2}{n}}$ and $\|v^{\star}\|_{\infty} \lesssim \frac{\log n}{k}$

Sparse PCA: simulations

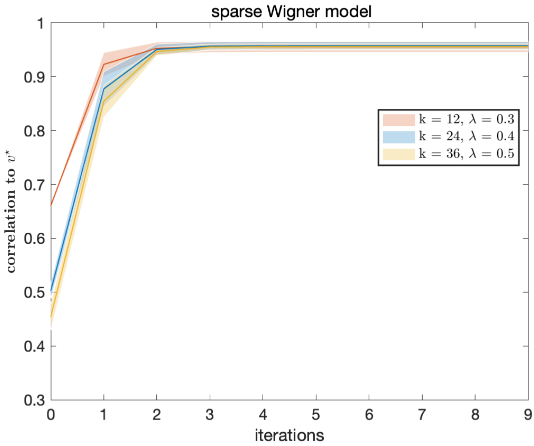

Figure: Convergence of AMP with diagonal maximization for different signal strengths with $n = 10000$. Repeat 40 times.

Auxiliary details

Define $\zeta_k := \left(\frac{\sqrt{2}}{2} - 1\right) z_k z_k^\top W_k z_k + \sum_{i=1}^{k-1} g_i^k z_i$

$$W_k z_k + \zeta_k = \phi_k \overset{\text{i.i.d}}{\sim} \mathcal{N}(0, \frac{1}{n} I_n)$$

- conditioning on $x_1, \{z_i\}_{i<k}$, W_k is a Wigner matrix in subspace $U_{k-1}^\perp$
- $W_k z_k$ has zero variance along the directions of $\{z_i\}_{i<k}$ and $\frac{2}{n}$ variance along the direction of z_k

Conditioning technique

AMP updates $x_{t+1} = Wm_t - \gamma_t m_{t-1}$

where $m^t = \eta_t(x_t), \quad \gamma_t = \langle \eta_t'(x_t) \rangle$

- $m_{-1} = 0, x_0 = 0$ and $x_1 = W\eta_t(0)$
- σ-algebra $\mathcal{F}_t$ generated by $\{x_0, x_1, \ldots, x_t\}$, conditioning on $\mathcal{F}$ is equivalent to conditioning on event

$$\mathcal{E}_t := \Big\{ x_1 + \gamma_0 m_{-1} = Wm_0,\ x_2 + \gamma_1 m_1 = Wm_1, \ldots,\ x_t + \gamma_{t-1} m_{t-1} = Wm_{t-1} \Big\}$$

- W conditioning on linear observations

$$W|_{\mathcal{F}_t} \stackrel{\mathsf{d}}{=} \mathbb{E}[W|_{\mathcal{F}_t}] + P_t^{\perp} W^{\mathsf{new}} P_t^{\perp}$$

$$W|_{\mathcal{F}_t} m^t \stackrel{\mathsf{d}}{=} \underbrace{W^{\mathsf{new}} P_t^{\perp} m^t}_{\text{Gaussian term}} + \underbrace{W^{\mathsf{new}} (I - P_t^{\perp}) m^t + \mathbb{E}[W|\mathcal{F}_t] m^t}_{\text{non-Gaussian term}}$$

Bolthausen (2006), Bayati & Montanari (2011), Rush & Venkataramanan (2016), Berthier, Montanari & Nguyen (2020)