Sample Spaces and (Un)conditional Probability **Notes 01**

Associated Reading: Wackerly 7, Chapter 2, Sections 1-4 and 7-10

We'll start with a general question: *What is probability?*

objective prob: the long-run frequency of occurrence of some "event"

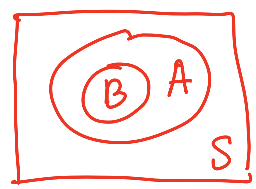
(e.g. heads in coin flips)

Subjective prob: one's belief in the rate of occurrence of an event.

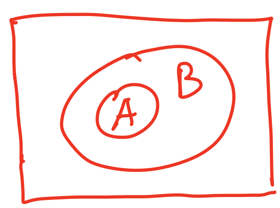
To begin our discussion of probability, we start with a review of set notation.

To match the book, we will:

- denote sets with capital letters: $A, B, C, \cdots$;
- denote the universal set (or the superset of all sets) with $\mathcal{S}$; and
- denote the null or empty set with $\emptyset$.



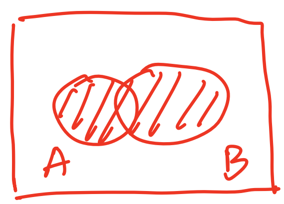
$A \supset B$



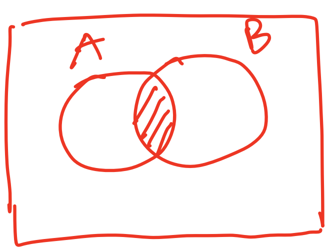
$A \subset B$

Term	Notation	Intuitive Terminology
superset	$A \supset B$	"encompasses"
subset	$A \subset B$	"within"
union	$A \cup B$	"or"
intersection	$A \cap B$	"and"
complement	$\bar{A}$	"not"

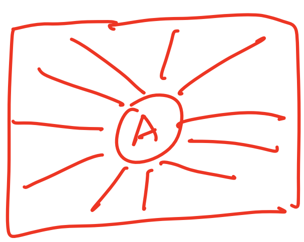
Let's draw Venn diagrams illustrating each of these terms:



$A \cup B$.



$A \cap B$.



$\bar{A}$ or A^c

Note the international standards for set notation on `http://en.wikipedia.org/wiki/ISO_31-11`; in particular, do not use logic notation on homeworks and tests (e.g., use $A \cap B$, not $A \wedge B$).

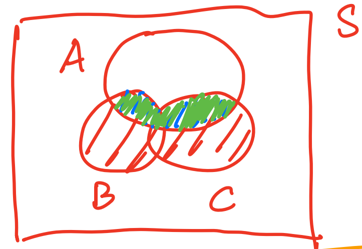
A few more important definitions and we are finished:

- $A \cup \bar{A} = \mathcal{S}$
- $A \cap B = \emptyset \;\Rightarrow$ A and B are *mutually exclusive* or *disjoint*
- the distributive and associative laws:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

prove



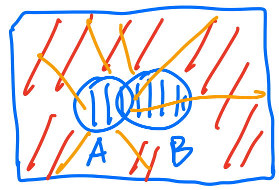
$$A \cap (B \cap C) = (A \cap B) \cap (A \cap C)$$

$$A \cup (B \cup C) = (A \cup B) \cup (A \cup C).$$

- De Morgan's laws:

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$



→ **EXAMPLE**: Wackerly 7, Exercise 2.5

a) $\boxed{A} = \boxed{(A \cap B) \cup (A \cap \bar{B})}$

$$= A \cap (\underbrace{B \cup \bar{B}}_{S}) = A$$

c) $\boxed{(A \cap B) \cap (A \cap \bar{B})} = \phi.$

$$= A \cap (\underbrace{B \cap \bar{B}}_{\Phi}).$$

$$= \Phi$$

b) if $B \subset A$

$$A = B \cup (A \cap \bar{B})$$

$\downarrow$ (a)

$$(A \cap B) \cup (A \cap \bar{B}) = B \cup (A \cap \bar{B})$$

bc $B \subset A$: $A \cap B = B$

So: *What is an experiment?*

A process by which observations are made.

active. you control an apparatus and its settings and collect data.

passive: you observe and collect data.

→ **EXAMPLE**: A person tosses two coins. What are the possible outcomes? (Note that we will use this example to define some terminology.)

$S = \{ HH, \underbrace{HT, TH, TT} \}$

sample space

a simple even : cannot be decomposed.

compound event (e.g. "at least one tail") can be decomposed

→ **EXAMPLE**: What is the sample space if

(a) a player shoots free throws until she misses?

S : success

F : failure.

$S = \{ F, SF, SSF, SSSF, \ldots \}$.

infinite.

(b) a person reads any two of three books A, B, and C?

$S = \{ AB, AC, BC \}$.

"BA" is the same as "AB"

order may or may not matter depending on experiment definition.

In the example above, what are the relative frequencies of each of the listed events?

don't know : more info needed.

→ **EXAMPLE**: Rice (2nd edition), Problem 1.8.1:

A coin is tossed three times and the sequence of heads and tails is recorded.

(a) List the sample space.

(b) List the elements that make up the following events: (1) A = at least two heads, (2) B = the first two tosses are heads, (3) C = the last toss is a tail.

(c) List the elements of the following events. (1) $\overline{A}$, (2) $A \cap B$, (3) $A \cup C$.

a) $S = \{ HHH, HHT, HTH, HTT, THH, THT, TTT, TTH \}$

b) $A = \{ HHH, HHT, HTH, THH \}$

$B = \{ HHT, HHH \}$.

$C = \{ HHT, HTT, THT, TTT \}$.

c). $\overline{A} = \{ HTT, THT, TTH, TTT \}$.

$A \cap B = \ldots$

$A \cup C = \ldots$

So: *what do we know about relative frequencies of events?*

① must be ≥ 0 and ≤ 1.

② the frequency of S is 1.

③ the frequency of a compound event equals to sum of the frequencies of its constituent simple events.

Let $\mathcal{S}$ be a sample space. A *probability measure* on $\mathcal{S}$ is a function P from subsets of $\mathcal{S}$ to $\mathbb{R}$ that satisfies the following *axioms*:

P: subset of $S \longrightarrow \mathbb{R}$.

(1) if $A \subset S$ $P(A) \geq 0$ $P(A) \leq 1$.

(2) $P(S) = 1$.

(3) if $A_1\ A_2 \cdots A_K$ are disjoint events

then $P(A_1 \cup A_2 \cdots \cup A_K) = P(A_1) + P(A_2) + \cdots + P(A_K) = \sum_{i=1}^{K} P(A_i)$

→ **EXAMPLE**: Use the axioms of probability to express $P[A]$ as a function of $P[\bar{A}]$.

$A \cup \bar{A} = S$ $A \cap \bar{A} = \phi$ (disjoint)

$1 = P(S) = P(A \cup \bar{A}) = P(A) + P(\bar{A}) \Rightarrow P(A) = 1 - P(\bar{A})$.

→ **EXAMPLE**: Wackerly 7, Problem 2.15

2.15 An oil prospecting firm hits oil or gas on 10% of its drillings. If the firm drills two wells, the four possible simple events and three of their associated probabilities are given in the accompanying table. Find the probability that the company will hit oil or gas

a on the first drilling and miss on the second.

b on at least one of the two drillings.

Simple Event	Outcome of First Drilling	Outcome of Second Drilling	Probability
E_1	Hit (oil or gas)	Hit (oil or gas)	.01
E_2	Hit	Miss	?
E_3	Miss	Hit	.09
E_4	Miss	Miss	.81

A B.

	B	$\bar{B}$	
A	$P(A \cap B)$	$P(A \cap \bar{B})$	$P(A)$
	+	+	
$\bar{A}$	$P(\bar{A} \cap B)$	$P(\bar{A} \cap B)$	$P(\bar{A})$
	‖	‖	
	$P(B)$	$P(\bar{B})$	1

denote H_1 : Hit oil on the first drilling.

$\bar{H_1}$: doesn't hit oil on - - - -

H_2 : Hit oil on the second - - -

$\bar{H_2}$: - - - -

	H_2	$\bar{H_2}$
H_1	0.01	?
$\bar{H_1}$	0.09	0.81

$P(H_1 \cap \bar{H_2})$

$= 1 - 0.01 - 0.09 - 0.81 = 0.09$.

$P(A \cap B) + P(\bar{A} \cap B)$

$\overset{?}{=} P((A \cap B) \cup (\bar{A} \cap B))$

$= P(B \cap (A \cup \bar{A}))$

$= P(B)$

$P(\text{"at least one hit"}) = 1 - P(\bar{H}_1 \cap \bar{H}_2) = 1 - 0.81 = 0.19.$

The book demonstrates two techniques for computing probabilities, the *sample-point* method and the *event-composition* method. The former utilizes counting methods (combinations and permutations) that do not generally arise in real-life analysis situations (outside of, say, gambling, which are among the things in life that we Do Not Officially Condone™).[a] So we will not cover it. We will come back to event decomposition on a future slide . . . but before that, we will cover *conditional probability*.

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$\propto$

The probability of an event (e.g., A) may depend on whether other events (e.g., B) occur. If so, we compute conditional probabilities, with conditions placed to the right of a vertical bar (e.g., $P[A|B]$).

When parsing word problems, one can easily identify when the probability is conditional: the words **if** or **given** will generally be used. For instance:

- What is the probability of selecting a red ball in the second draw from a jar of m black balls and n red balls, **if** a black ball is drawn first? (Sampling here is done without replacement.)
- What is the probability of being a French speaker **given** that one lives in Brussels?

F : "speaks French" $\quad P(F) = \frac{3}{70}$

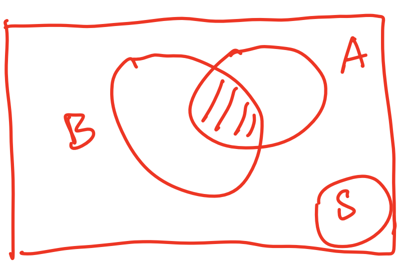
B : "lives in Brussels" $\quad P(B) = \frac{1}{7000}$

$P(F|B) \geq \frac{3}{5} \qquad P(B|F) = \frac{1}{5000} \qquad$ ② $P(F|B)$

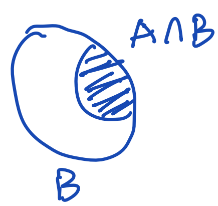
① . $P(F) \neq P(F|B)$ unconditional × conditional prob. $\neq P(B|F)$.

Important: in conditional probability, *causality is not implied!* In other words, event A does not have to follow B. B simply has to occur at some point.

The following will help you to see intuitively how the conditional probability is computed:



$P(A|B)$.



$P(A|B) = \frac{P(A \cap B)}{P(B)}$

$P(B|A) = \frac{P(A \cap B)}{P(A)}$

→ **EXAMPLE:** Given the following 2×2 table of experimental outcomes, compute $P[A|B]$. Does it equal $P[B|A]$?

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{2/(2+6+1+9)}{3/(2+6+1+9)} = \frac{2}{3}.$

A∩B.

	B	$\bar{B}$
A	2	6
$\bar{A}$	1	9

8

3

$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{2/18}{8/18} = \frac{1}{4}.$

→ **EXAMPLE:** If $P[A] = 0.5$, $P[B] = 0.3$, and $P[A \cap B] = 0.1$, what is $P[A|A \cup B]$ and $P[A|A \cap B]$?

- $P(A|A\cup B) = \dfrac{P(A \cap (A \cup B))}{P(A\cup B)} = \dfrac{P(A)}{P(A\cup B)} = \dfrac{0.5}{0.7} = \dfrac{5}{7}.$

	B	$\overline{B}$	
A	0.1	0.4	0.5
$\overline{A}$	0.2	0.3	
	0.3		

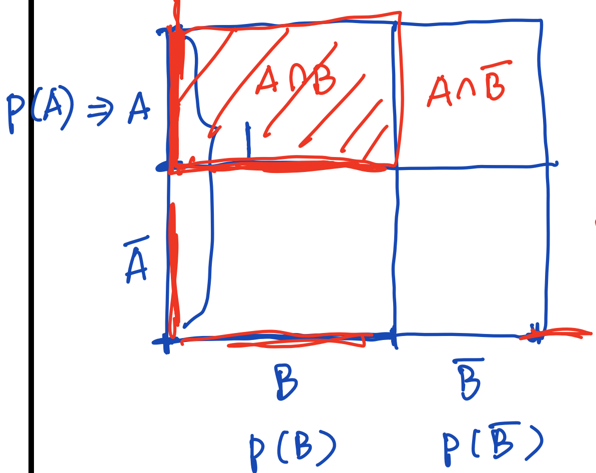
- $P(A|A\cap B) = \dfrac{P(A \cap (A \cap B))}{P(A\cap B)} = \dfrac{P(A\cap B)}{P(A\cap B)} = 1$

The events A and B are said to be *independent* if any of the following conditions hold:

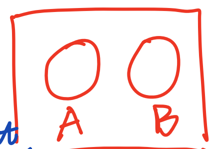
- $P[A|B] = P[A]$;
- $P[B|A] = P[B]$; and/or
- $P[A \cap B] = P[A]P[B]$.

$P(A|B) = \dfrac{P(A\cap B)}{P(B)} = P(A) \Leftrightarrow P(A\cap B) = P(A)\cdot P(B).$

The following is an example of independence as rendered on a Venn diagram:



if A, B are independent.



$P(A\cap B) = 0.$

two events disjoint.

$\boxed{P(A\cap B) = P(A)\cdot P(B)}$

$P(A\cap B) = P(A)\cdot P(B)$

$\Downarrow$

two events are independent

if (A, B) indep events

so as $(\overline{A}\ B)$

$(\overline{B}\ A)$

$(\overline{A}\ \overline{B})$.

use area to denote probability of event.

Now that we've learned the concepts of conditional probabilities and independence, we can write down two important laws of probability.

→ **THEOREM 2.5.** $\boxed{\text{The Multiplicative Law:}}$ the probability of the intersection of two events A and B is given by:

$$\boxed{P[A\cap B]} = P[A]P[B|A] = P[B]P[A|B]$$

$$= 0 \ \text{ if } A \text{ and } B \text{ are disjoint}$$

$$= P[A]P[B] \ \text{ if } A \text{ and } B \text{ are independent}$$

$$P(B|A) = \frac{P(A\cap B)}{P(A)}.$$

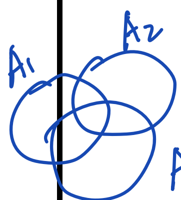
We can generalize this result to n events $\{A_1, \cdots, A_n\}$:

$$P(A_1\cap \underbrace{A_2 \cdots \cap A_n}_{B}) = \boxed{P(\underbrace{A_2\cap A_3 \cdots \cap A_n}_{B})}\, P(A_1 | \underbrace{A_2\cdots\cap A_n}_{B})$$

$$= \boxed{P(\underbrace{A_3\cap A_4\cdots A_n}_{C})\cdot P(A_2 | \underbrace{A_3\cdots A_n}_{C})}\cdot P(A_1|A_2\cdots A_n)$$

$$\vdots$$

$$= \boxed{P(A_n)}\cdot\boxed{P(A_{n-1}|A_n)}\cdot\boxed{P(A_{n-2}|A_{n-1}\cap A_n)}\cdots P(A_1|A_2\cap\cdots A_n)$$



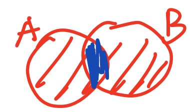
→ **THEOREM 2.6.** The Additive Law: the probability of the union of two events A and B is

$$P(A_1\cup A_2 \cdots \cup A_n)$$

$$\boxed{P[A\cup B]} = \boxed{P[A] + P[B] - P[A\cap B]}$$

$$= P[A] + P[B] \ \text{ if } A \text{ and } B \text{ are disjoint}$$

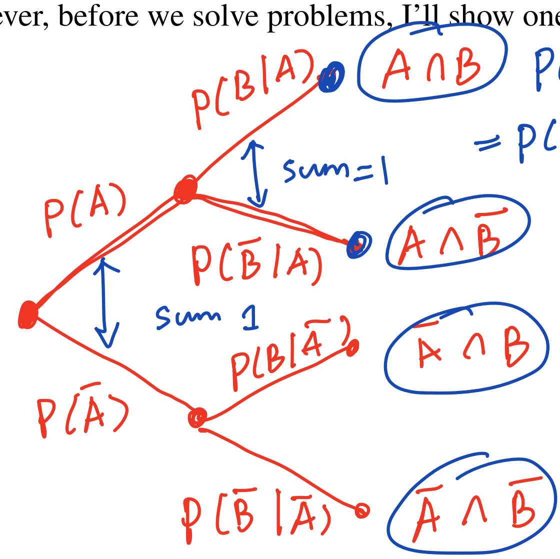
$$= P[A] + P[B] - P[A]P[B] \ \text{ if } A \text{ and } B \text{ are independent}$$



Now, after stating one more result, we'll have sufficient tools at our disposal to extend our probabilistic modeling abilities beyond the sample-point method:

→ **THEOREM 2.7.** If A is an event, then $P[A] = 1 - P[\bar{A}]$.

However, before we solve problems, I'll show one more tool for probabilistic modeling, the *decision tree*:



$$P(A\cap B) = P(A)\cdot P(B|A)$$

$$P(B|A) + P(\bar{B}|A) = 1$$

$$= \frac{P(A\cap B)}{P(A)} + \frac{P(A\cap\bar{B})}{P(A)}.$$

$$= \frac{P(A\cap B) + P(A\cap\bar{B})}{P(A)}$$

$$= \frac{P((A\cap B)\cup(A\cap\bar{B}))}{P(A)}$$

$$= \frac{P(A)}{P(A)} = 1.$$

√ $A\cap B \Rightarrow$ a simple event.

√ $A \Rightarrow$ not a simple event

decompose $A\cap B$ $\quad A\cap\bar{B}$

The *event-composition* method presented in Section 2.9 of Wackerly 7 can be boiled down to:

1. When faced with a problem, define your events (e.g., F = "a crashed plane was found" and B = "the plane had an emergency locator beacon").
2. Write down what you know (both unconditional and conditional probabilities). Be careful to parse the problem correctly, i.e., to write any conditional probabilities in the correct "order" (e.g., "if a plane has a locator beacon, there is a 90% chance it will be found after a crash" $\Rightarrow P[F|B] = 0.9 \neq P[B|F]$).
3. Write down what quantity you want to solve for. $P(F \wedge B)$
4. Link the items in (2) with the desired result in (3) via the laws of probability, in any way possible.

The book itself says "the best way to learn how to solve probability problems is to learn by doing," so let's go ahead and do . . .

→ **EXAMPLE** (courtesy O. Meyer): the information you get with a certain prescription drug states:

- There is a 10% chance of experiencing headaches (denoted H). $P(H) = 0.1$
- There is a 15% chance of experiencing nausea (denoted N). $P(N) = 0.15$
- There is a 5% chance of experiencing both side effects (i.e., $H \cap N$). $P(H \wedge N) = 0.05$.

(a) Are the events H and N disjoint? (b) What is the probability of experiencing at least one of the two side effects? (c) What is the probability of experiencing exactly one of the side effects? (d) If you experience nausea, what is the probability that you'll also experience headaches?

a). $P(H \wedge N) = 0.05 \neq 0$

Not disjoint.

b). $P(\text{at least one of two side effects})$

$= P(H \cup N)$

$= P(H) + P(N) - P(H \wedge N)$.

$= 0.2$.

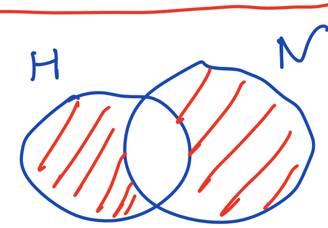
c) $P(H \wedge \bar{N}) + P(N \wedge \bar{H})$.

$= 0.05 + 0.1$

$= 0.15$.

$P(H | N)$.

$$= \frac{P(H \wedge N)}{P(N)} = \frac{0.05}{0.15}$$



$P(\text{shaded}) = P(H \cup N) - P(H \wedge N)$

$= 0.2 - 0.05 = 0.15$.

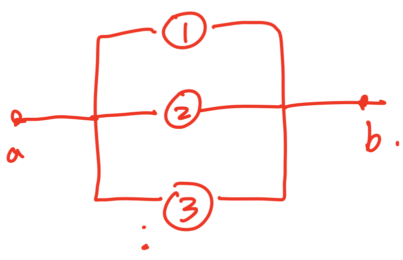
	N	$\bar{N}$	
H	0.05	0.05	0.1
$\bar{H}$	0.1		
	0.15		

→ **EXAMPLE:** Wackerly 7, Exercise 2.97.

2.97 Consider the following portion of an electric circuit with three relays. Current will flow from point *a* to point *b* if there is at least one closed path when the relays are activated. The relays may malfunction and not close when activated. Suppose that the relays act independently of one another and close properly when activated, with a probability of .9.

a What is the probability that current will flow when the relays are activated?

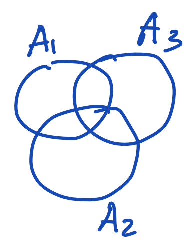
✓ **b** Given that current flowed when the relays were activated, what is the probability that relay 1 functioned?



A_i : relay i works well.

$P(A_i) = 0.9$.

A_1 A_2 A_3 are indep.



a). $P(\text{will flow}) = P(A_1 \cup A_2 \cup A_3)$

$= \underset{\text{additive law}}{\cdots}$ (exercise!).

$= 1 - P(\underbrace{\overline{A_1 \cup A_2 \cup A_3}}_{\text{all fail}})$

What if you have 10 relays? (exercise!!) ⇒

$= 1 - P(\bar{A_1} \cap \bar{A_2} \cap \bar{A_3})$. independent.

$= 1 - P(\bar{A_1}) \cdot P(\bar{A_2}) \cdot P(\bar{A_3})$

$= 1 - 0.1 \cdot 0.1 \cdot 0.1 = 0.999$.

b) $P(A_1 \mid A_1 \cup A_2 \cup A_3) = \dfrac{P(\boxed{A_1 \cap (A_1 \cup A_2 \cup A_3)})}{P(A_1 \cup A_2 \cup A_3)}$ ⇑ A_1

$= \dfrac{P(A_1)}{P(A_1 \cup A_2 \cup A_3)} = \dfrac{0.9}{0.999}$

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$$P(M_2 \mid M_1) = \frac{P(M_1 \cap M_2)}{P(M_1)}.$$ ✓

→ **EXAMPLE:** Wackerly 7, Exercise 2.101.

2.101 Articles coming through an inspection line are visually inspected by two successive inspectors. When a defective article comes through the inspection line, the probability that it gets by the first inspector is .1. The second inspector will "miss" five out of ten of the defective items that get past the first inspector. What is the probability that a defective item gets by both inspectors?

let M_1: inspector 1 gets the article. $P(M_1) = 0.1$

M_2: inspector 2 gets the article. $P(M_2 \mid M_1) = \frac{1}{2}$.

$$P(M_1 \cap M_2) = \underbrace{P(M_1)}_{0.1} \cdot \underbrace{P(M_2 \mid M_1)}_{0.5}. = 0.05.$$

A useful thing to have at your fingertips are the probabilities for each outcome of rolling two fair six-sided dice:

$P(\text{sum of two dice} = 6) = \frac{5}{36}$.

Roll	2	3	4	5	6	7	8	9	10	11	12
Probability	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

→ **EXAMPLE:** Wackerly 7, Exercise 2.119(a).

***2.119** Suppose that two balanced dice are tossed repeatedly and the sum of the two uppermost faces is determined on each toss. What is the probability that we obtain

✓ **a** a sum of 3 before we obtain a sum of 7?

b a sum of 4 before we obtain a sum of 7?

3 | 7

$$\frac{\frac{2}{36}}{\frac{2}{36} + \frac{6}{36}} = \frac{1}{4}$$

$P(\text{sum 3 comes before sum 7})$.

S_i: seeing "3" on throw i

F_i: not "3" or "7" on throw i

W: "3" before "7".

$$W = \{\underbrace{S_1, F_1S_2, F_1F_2S_3, F_1F_2F_3S_4, \ldots}\}$$

$= P(W)$

$= P(S_1) + P(F_1S_2) + P(F_1F_2S_3) + \ldots$

$P(F_1S_2) = P(F_1) \cdot P(S_2)$

$P(F_1F_2S_3) = P(F_1) \cdot P(F_2) \cdot P(S_3)$

$$= \frac{2}{36} + \underbrace{\left(1 - \frac{2}{36} - \frac{6}{36}\right)}_{28/36} \cdot \frac{2}{36}.$$

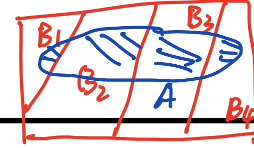
$$+ \frac{28}{36} \cdot \frac{28}{36} \cdot \frac{2}{36} + \left(\frac{28}{36}\right)^3 \cdot \frac{2}{36} + \left(\frac{28}{36}\right)^4 \cdot \frac{2}{36} + \cdots$$

$$= \frac{2}{36}\left[\underbrace{1 + \frac{28}{36} + \left(\frac{28}{36}\right)^2 + \left(\frac{28}{36}\right)^3 + \cdots}\right]$$

$(x < 1)$

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

$$= \frac{2}{36} \cdot \frac{1}{1 - \frac{28}{36}} = \frac{1}{4}$$



Assume that $\{B_1, \cdots, B_k\}$ is a *partition* of the sample space $\mathcal{S}$, i.e., that

$$\mathcal{S} = B_1 \cup \cdots \cup B_k, \text{ with } B_i \cap B_j = \emptyset \; \forall \, i \neq j.$$

(The B_i's need not be simple events.) Assuming $P[B_i] > 0 \; \forall \, i$, then we can write down . . .

→ **THEOREM 2.8.** Law of Total Probability (LoTP): for any event A

$$P[A] = \sum_{i=1}^{k} \boxed{P[A|B_i]P[B_i]} \text{ and}$$

$P(A \cap B_i)$.

$P(A) = P(A \cap S)$
$= P(A \cap (\cup B_i))$
$= P(\bigcup_{i=1}^{k} (A \cap B_i))$.
$= \sum_{i=1}^{k} P(A \cap B_i)$.

→ **THEOREM 2.9.** Bayes' Rule: the conditional probability of each event in the partition of $\mathcal{S}$ is

$$P[B_j|A] = \frac{P[A|B_j]P[B_j]}{\sum_{i=1}^{k} P[A|B_i]P[B_i]} = \frac{P[A|B_j]P[B_j]}{P[A]}$$

→ **EXAMPLE:** Show $P[A \cap C|B] + P[A \cap \bar{C}|B] = P[A|B]$.

$P(A \cap C | B) + P(A \cap \bar{C} | B)$

$= \dfrac{P((A \cap C) \cap B)}{P(B)} + \dfrac{P((A \cap \bar{C}) \cap B)}{P(B)}$

$= \dfrac{P(A \cap C \cap B) + P(A \cap \bar{C} \cap B)}{P(B)} = \dfrac{P(A \cap B)}{P(B)} = P(A|B)$.

→ **EXAMPLE:** You are diagnosed with a disease, which has two types, A and $\bar{A}$. In the population at large, the probability of having types A and $\bar{A}$ are 10% and 90%, respectively. You undergo a test that is 80% accurate (i.e., if you have type X, the test will indicate you have type X 80% of the time, and the other type 20% of the time). The test indicates that you have type A. Do you immediately start treatment for type A?

$P(A) = 10\%$ $\quad P(\bar{A}) = 90\%$.

$P(\text{Test gives } A \mid A) = 80\%$

$P(\text{Test gives } \bar{A} \mid \bar{A}) = 80\%$.

define a test says A to be T

$\boxed{P(A|T)} = \dfrac{P(T \cap A)}{P(T)} = \dfrac{P(A)\,P(T|A)}{P(A)P(T|A) + P(\bar{A})P(T|\bar{A})}$

0.1, 0.8; 0.1 · 0.8, 0.9, 0.2

LoTP

$P(T) = P(A \cap T) + P(\bar{A} \cap T) = P(A) \cdot P(T|A) + P(\bar{A}) \cdot P(T|\bar{A})$.

partition $\{A, \bar{A}\}$

$= \dfrac{4}{13}$.

$P(\bar{A}|T) = \dfrac{9}{13}$.

→ The Monty Hall Problem

The Monty Hall Problem is named for the long-time host of the game show *Let's Make a Deal*. The simple version goes as follows: you are shown three doors; behind two of the doors are goats and behind the other is a car. You choose a door (say Door #1). Monty Hall then opens, say, Door #3 to reveal a goat, and asks you if you want to switch to Door #2.

So: do you stick with Door #1 or switch to Door #2?

P(see a car behind Door #1 | Monty opens door #3).

Define O_i: Monty opens door i.

C_i: car is behind door i.

$$\boxed{P(C_1 \mid O_3)} = \frac{P(C_1)\,P(O_3 \mid C_1)}{P(C_1)\,P(O_3 \mid C_1) + P(C_2)\,P(O_3 \mid C_2) + P(C_3)\,P(O_3 \mid C_3)}.$$

$\Rightarrow P(C_1 \cap O_3)$.

$\Rightarrow P(O_3)$

$P(C_i) = \frac{1}{3}$.

$P(O_3 \mid C_1) = \frac{1}{2}$

$P(O_3 \mid C_2) = 1$

$P(O_3 \mid C_3) = 0$

$$= \frac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{2}\cdot\frac{1}{3} + 1\cdot\frac{1}{3} + 0} = \frac{1}{3}$$

Since $P(C_1 \mid O_3) + P(C_2 \mid O_3) + \boxed{\overset{0}{P(C_3 \mid O_3)}} = 1$.

$$\Rightarrow \boxed{P(C_2 \mid O_3) = 1 - \frac{1}{3} = \frac{2}{3}.}$$

Switch !