Distributions of Functions of Random Variables **Notes 07**

Associated Reading: Wackerly 7, Chapter 6, Sections 1-4

Let's start by establishing the point of this chapter. You've conducted an experiment which yields observed data $Y_1, \cdots, Y_n$, and now you need to analyze these data. What do you do?

1. You determine what property of the underlying conceptual population that you want to statistically infer. (For instance, your Y's may all be drawn from some known or unknown distribution whose mean is μ, and you want to use your data to infer μ.)
2. You select an *estimator* of the property you want to infer. (For instance, you might select the sample mean as an estimator of the population mean.)
3. As your chosen estimator is a function of the random variables Y, it itself is a random variable, sampled from some distribution. In order to perform precise statistical inference, you need to determine this distribution. Not just its mean, or its mean and variance, but its actual shape. (Up until now, we've fallen back on imprecise inference using Tchebysheff's Theorem. No more!)

Here's a motivating example:

① observe $Y_1\ Y_2 \cdots Y_n \overset{iid}{\sim} P$ (P: some distribution of interest).

② We wish to infer mean of population: μ_P

$\Rightarrow$ estimator $\bar{Y} = \dfrac{Y_1 + \cdots + Y_n}{n} = \dfrac{1}{n}\sum_{i=1}^{n} Y_i$.

③ distribution of $\bar{Y} \neq P$.

$\bar{Y}$ is a r.v. has distribution (call it Q.)

④ we can't compute precisely $P(\mu_Q - \sigma_Q \le \bar{Y} \le \mu_Q + \sigma_Q)$.

without knowing dist Q.

$\Rightarrow$ general question: what is the distribution $g(Y_1 \cdots Y_n)$?

What we will look at in Chapter 6 are three methods that one can use to try to determine the pmf/pdf of an estimator (or any other function of random variables, for that matter):

- ✓ the *method of distribution functions*; cdf $\to$ pdf
- ✓ the *method of transformations* (which is really just a simplified form of the preceding method); and
- the *method of moment-generating functions* (or *mgfs*). $\Rightarrow$ Note 8

In this notes set, we'll work through examples of the first two methods, and in the next set of notes, we'll turn to mgfs.

dist of $g(Y)$ or $g(Y_1\ Y_2 \cdots Y_n)$

The method of distribution functions applies methods that you have already learned up to now in this class. The main issue with it is its apparent complexity. I'll break it down into steps here:

① Goal: identify $U = g(Y)$ or $g(Y_1, Y_2)$

② define cdf U as

$$\boxed{F_U(u) = P(U \le u) = P(g(\cdot) \le u).}$$

(i) if $U = g(Y)$ $\quad P(g(Y) \le u) = \sum_{y:\, g(y) \le u} p(y)$ or $\int_{y:\, g(y) \le u} f(y)\, dy$

(ii) if $U = g(Y_1, Y_2)$ $\quad P(g(Y_1,Y_2) \le u) = \sum_{y_1, y_2:\, g(y_1,y_2) \le u} p(y_1,y_2)$ or $\int_{y_1,y_2:\, g(y_1,y_2) \le u} f(y_1,y_2)\, dy_1 dy_2$

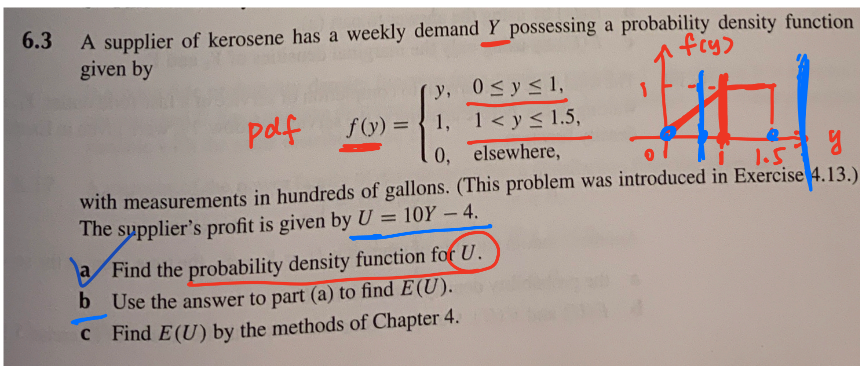
③ Given $F_U(u)$ (cdf of U).

$\boxed{\text{pdf}}$ $\quad f_U(u) = \dfrac{dF_U}{du}$

$\boxed{\text{pmf.}}$ $\quad$ find jump point of F_U $\Downarrow$ table.

To internalize these details, there is, as usual, no substitute to working through problems.

→ **EXAMPLE.** Wackerly 7, Exercise 6.3(a,b)


6.3 A supplier of kerosene has a weekly demand Y possessing a probability density function given by

$$f(y) = \begin{cases} y, & 0 \le y \le 1, \\ 1, & 1 < y \le 1.5, \\ 0, & \text{elsewhere,} \end{cases}$$

with measurements in hundreds of gallons. (This problem was introduced in Exercise 4.13.) The supplier's profit is given by $U = 10Y - 4$.

a Find the probability density function for U.

b Use the answer to part (a) to find $E(U)$.

c Find $E(U)$ by the methods of Chapter 4.

b) $E[U] = \int u f_U(u)\, du.$

$$= \int_{-4}^{6} u \frac{u+4}{100}\, du + \int_6^{11} u \cdot \frac{1}{10}\, du = 5\frac{7}{12}.$$

Since $U = 10Y - 4$

$E U = E[10Y - 4]$

$= 10 E[Y] - 4.$

a). $U = 10Y - 4$

method of dist function.

cdf $P(U \le u) = P(10Y - 4 \le u) = P\left(Y \le \dfrac{u+4}{10}\right).$ ≥ 0

Let divide into cases.

Case 1: $\dfrac{u+4}{10} \le 0$ $\quad P[U \le u] = 0$

Case 2: $\dfrac{u+4}{10} \in [0,1].$ $\quad P\left(Y \le \dfrac{u+4}{10}\right) = \int_0^{\frac{u+4}{10}} y\, dy = \dfrac{(u+4)^2}{200}$ $\quad 1 \Rightarrow u = 6$

Case 3: $\dfrac{u+4}{10} \in [1, 1.5]$ $\quad P\left(Y \le \dfrac{u+4}{10}\right) = P(Y \le 1) + P\left(1 \le Y \le \dfrac{u+4}{10}\right).$

$$= \frac{10^2}{200} + \int_1^{\frac{u+4}{10}} 1\, dy = \frac{u-1}{10}$$

Set $u = 6$ in Case 2.

Case 4: $\dfrac{u+4}{10} > 1.5$ $\quad P\left(Y \le \dfrac{u+4}{10}\right)$ $\quad 1$

$F_U(u) = \begin{cases} 0 & u \le -4 \\ \dfrac{(u+4)^2}{\ldots} & -4 < u \le 6 \end{cases}$ $\Rightarrow$ $f_U(u) = \begin{cases} 0 & u \le -4 \\ \dfrac{u+4}{\ldots} & u \in [-4, 6] \end{cases}$

$$\begin{cases} \frac{u}{200} & \\ \frac{u-1}{10} & 6 < u < 11 \\ 1 & u \geq 11 \end{cases}$$

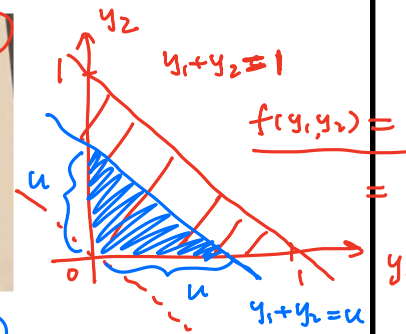
$\frac{1}{10}$ $u \in (6, 11)$

0 $u \geq 11$

→ **EXAMPLE.** Wackerly 7, Exercise 6.9(a,b)

6.9 Suppose that a unit of mineral ore contains a proportion Y_1 of metal A and a proportion Y_2 of metal B. Experience has shown that the joint probability density function of Y_1 and Y_2 is uniform over the region $0 \leq y_1 \leq 1$, $0 \leq y_2 \leq 1$, $0 \leq y_1 + y_2 \leq 1$. Let $U = Y_1 + Y_2$, the proportion of either metal A or B per unit. Find

- **a** the probability density function for U.
- **b** $E(U)$ by using the answer to part (a).
- **c** $E(U)$ by using only the marginal densities of Y_1 and Y_2.



$f(y_1, y_2) = \frac{1}{\text{area}} = 2$

a) $U = Y_1 + Y_2$

method of distribution function.

$F_U(u) = P(U \leq u) = P(Y_1 + Y_2 \leq u)$

$= \iint_{\substack{y_1, y_2 \\ y_1 + y_2 \leq u}} f(y_1, y_2)\, dy_1\, dy_2.$

$= 2 \cdot \boxed{\iint_{y_1 + y_2 \leq u} 1\, dy_1\, dy_2} = 2 \cdot \text{area}(\ldots)$

$$= \begin{cases} u^2 & u \in [0, 1] \\ 0 & u < 0 \\ 1 & u > 1 \end{cases}$$

cdf $\Rightarrow f_U(u) = \begin{cases} 2u & 0 \leq u \leq 1 \\ 0 & \text{otherwise} \end{cases}$

b) $\mathbb{E}\,U$

$= \int u \cdot f(u)\, du$

$= \int_0^1 u \cdot 2u\, du = \frac{2}{3}.$

$\mathbb{E}\,U = \mathbb{E}\,Y_1 + \mathbb{E}\,Y_2.$

→ **EXAMPLE.** Wackerly 7, Exercise 6.7

6.7 Suppose that Z has a standard normal distribution.

- **a** Find the density function of $U = Z^2$.
- **b** Does U have a gamma distribution? What are the values of α and β?
- **c** What is another name for the distribution of U?

$Z \sim N(0, 1)$

$U = Z^2$

a) method of distribution function.

$N(0,1)$.

$\Phi(x) = P(X \leq x)$ $X \sim N(0,1)$ cdf.

$F_U(u) = P(U \leq u) = P(Z^2 \leq u) = P(-\sqrt{u} \leq Z \leq \sqrt{u})$

cdf of U

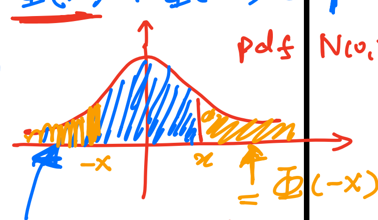
$= \Phi(\sqrt{u}) - \Phi(-\sqrt{u})$

$= \Phi(\sqrt{u}) - (1 - \Phi(\sqrt{u}))$

$= 2\Phi(\sqrt{u}) - 1.$ $u \geq 0$

$\Phi(x) + \Phi(-x) = 1$

pdf $N(0,1)$.



$= \Phi(-x)$

$x \geq 0$

$\Phi(x)$.

$f_U(u) = \frac{d F_U(u)}{du} = \frac{d(2\Phi(\sqrt{u}) - 1)}{du}$

Chain rule $\nearrow$ $= 2 \cdot \Phi'(\sqrt{u}) \cdot \frac{d\sqrt{u}}{du} = 2 f_Z(\sqrt{u}) \cdot \frac{1}{2} \frac{1}{\sqrt{u}} = \frac{1}{\sqrt{2\pi u}} e^{-\frac{u}{2}}$

density

$f_Z(\sqrt{u}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{u})^2}{2}}$

$u \geq 0$

Gamma $(\alpha = \frac{1}{2}, \beta = 2)$

Fun fact: Suppose Y: cdf $F_Y(y)$. Suppose $U \sim \text{Unif}(0,1)$

what is dist $F^{-1}(U)$?

when $\beta = 2$

$Z = F_Y^{-1}(U)$ $\quad F_Z(z) = P(Z \le z) = P(F_Y^{-1}(U) \le z) = P(U \le F_Y(z)) = F_Y(z)$ $\Rightarrow \chi^2(\nu = 2\alpha = 1)$.

$\Rightarrow$ Z has distribution same with Y.

There is another application of this methodology that is useful for simulating data. Let's say you want to simulate a datum Y from an arbitrary distribution. One way to do this is to simulate a datum U from a Uniform(0,1) distribution (which is easy to do given any random number generator), and then transform that datum such that $Y = g(U)$ is sampled from the distribution of your choice.

→ **EXAMPLE.** Wackerly 7, Exercise 6.15

6.15 Let Y have a distribution function given by

$$F(y) = \begin{cases} 0, & y < 0, \\ 1 - e^{-y^2}, & y \ge 0. \end{cases}$$

Find a transformation $G(U)$ such that, if U has a uniform distribution on the interval (0, 1), $G(U)$ has the same distribution as Y.

$F(y) = 1 - e^{-y^2}$ $\quad y \ge 0$.

let $U = F(Y) \Rightarrow Y \Rightarrow F^{-1}(U)$.

$U = 1 - e^{-Y^2}$

$\Downarrow$

$e^{-y^2} = 1 - U$

$\Downarrow$

$y = \sqrt{-\log(1-U)}$.

take $G(U) = \sqrt{-\log(1-U)}$

if we want to generate distribution whose cdf is F_Y. then,

① generate $U \sim Unif(0,1)$

② find F_Y^{-1}

③ $F_Y^{-1}(U)$. has cdf $= F_Y$.

$\propto$

The method of transformations is, as mentioned, a simplified version of the method of distribution functions that one can apply when the function $U = h(Y)$ is *strictly increasing* or *strictly decreasing* over the support of $f(y)$. For instance, if the support of $f(y)$ is the range $-1 \le y \le 1$ and $U = Y^2$, you cannot use the method of transformations, because $h(Y)$ decreases over the range $-1 \le y < 0$ and increases over the range $0 < y \le 1$. But if the support of $f(y)$ is the range $0 \le y \le 1$ and $U = Y^2$, you *can* use the method of transformations, as $h(Y)$ is strictly increasing.

The method of transformations is based on the following algorithm:

$U = h(Y)$ $\quad$ suppose $h \uparrow$

cdf $F_U(u) = P(U \le u) = P(h(Y) \le u) = P(Y \le h^{-1}(u)) = F_Y(h^{-1}(u))$.

pdf. $\boxed{f_U(u)} = \dfrac{\partial F_U(u)}{du} = f_Y(h^{-1}(u)) \cdot \dfrac{d h^{-1}(u)}{du}$. $\quad$ (assume $h \uparrow$).

Chain rule.

suppose $h \downarrow$

$F_U(u) = P(h(Y) \le u) = P(Y \ge h^{-1}(u)) = 1 - F_Y(h^{-1}(u))$.

pdf $f_U(u) = \dfrac{d F_U(u)}{du} = - f_Y(h^{-1}(u)) \cdot \dfrac{dh^{-1}(u)}{du}$ $\quad$ (*)

< 0

≥ 0 $\quad$ putting together: $f_U(u) = f_Y(h^{-1}(u)) \cdot \left| \dfrac{d h^{-1}(u)}{du} \right|$

Steps: ① determine $h^{-1}(u)$, confirm that h ...

② compute $\frac{d\,h^{-1}(u)}{du}$ ③ plug $h^{-1}(u)$, $\frac{d\,h^{-1}(u)}{du}$ into ⊛. Done ✓

→ **EXAMPLE.** Wackerly 7, Exercise 6.23(c)

6.23 In Exercise 6.1, we considered a random variable Y with probability density function given by

$$f(y) = \begin{cases} 2(1-y), & 0 \le y \le 1, \\ 0, & \text{elsewhere,} \end{cases}$$

and used the method of distribution functions to find the density functions of

a $U_1 = 2Y - 1$.

b $U_2 = 1 - 2Y$.

c $U_3 = Y^2$.

Use the method of transformation to find the densities of U_1, U_2, and U_3.

c). $U_3 = Y^2 = h(Y)$ $\boxed{h(x) = x^2}$

on $[0,1]$ h is strictly increasing.

method of transformations is applicable.

$h(Y): Y \to U = Y^2$

$h^{-1}(U): U \to Y$

Steps

①. $h^{-1}(u)$

$U = h(Y) = Y^2$

$\Rightarrow \boxed{Y = \sqrt{U}}$

$\boxed{h^{-1}(u)} = \sqrt{u}$ ✓

② $\frac{d\,h^{-1}(u)}{du} = \frac{1}{2} \cdot \frac{1}{\sqrt{u}}$ ✓

③ $f_U(u) = f_Y(h^{-1}(u)) \cdot \left| \frac{dh^{-1}}{du} \right|$

$= 2(1 - \sqrt{u}) \cdot \frac{1}{2\sqrt{u}}$

$= \begin{cases} \frac{1}{\sqrt{u}} - 1 & \text{when } u \in (0,1] \\ 0 & \text{otherwise.} \end{cases}$

→ **EXAMPLE.** Wackerly 7, Exercise 6.29

6.29 The speed of a molecule in a uniform gas at equilibrium is a random variable V whose density function is given by

$$f(v) = av^2 e^{-bv^2}, \quad v > 0,$$

where $b = m/2kT$ and k, T, and m denote Boltzmann's constant, the absolute temperature, and the mass of the molecule, respectively. $m > 0$

a Derive the distribution of $W = mV^2/2$, the kinetic energy of the molecule.

b Find $E(W)$.

a) $\boxed{W = m \cdot \frac{V^2}{2}}$ $V \sim f(v) = av^2 e^{-bv^2}$ $v > 0$.

① $W = h(V) = \frac{m}{2} \cdot V^2$

$\Rightarrow$ find $h^{-1}(w)$.

$W = \frac{m}{2} V^2 \Rightarrow V = \sqrt{\frac{2W}{m}}$ $V > 0$

② $\frac{d\,h^{-1}(w)}{dw} = \frac{d\left(\sqrt{\frac{2w}{m}}\right)}{dw} = \sqrt{\frac{2}{m}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{w}} = \frac{1}{\sqrt{2mw}}$

③ $f_W(w) = f_V(h^{-1}(w)) \cdot \left| \frac{d\,h^{-1}(w)}{dw} \right|$

$= a \cdot \left(\sqrt{\frac{2w}{m}}\right)^2 \cdot e^{-b \cdot \left(\sqrt{\frac{2w}{m}}\right)^2} \cdot \sqrt{\frac{1}{2mw}}$.

$b = \frac{m}{2kT}$ $= \begin{cases} \frac{\sqrt{2}a}{m^{3/2}} w^{\frac{1}{2}} e^{-\frac{w}{kT}} & w \in (0, +\infty) \\ 0 & \text{otherwise.} \end{cases}$

$\sim$ Gamma$\left(\alpha = \frac{3}{2}, \beta = kT\right)$

b) $E[W]$

$= \frac{3}{2} \cdot kT$

$\alpha \quad \beta$.

$E[W]$

$= E\left[\frac{mV^2}{2}\right]$

$= \frac{m}{2} E[V^2]$